On the order-enriched category of overlap-algebras

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References

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Inhabitedness and Overlap

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$$X \neq \emptyset \quad \text{and} \quad \textit{X is inhabited}$$

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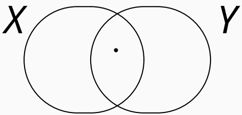
and, more generally, between $X \cap Y \neq \emptyset$ and



$$\equiv \quad X \between Y$$ (Sambin's notation)

i.e. $X \cap Y$ *is inhabited*.

The relation $\between$ is used in Sambin’s approach to topology (the Basic Picture) to express the link between interior and closure, to express continuity, etc.

Properties of $\between$

What are the constructive properties of the relation $\between$

$$X \between Y \Rightarrow Y \between X$$

$$\left.\begin{array}{l} X \between Y \\ Y \subseteq Z \end{array}\right\} \Rightarrow X \between Z$$

$$X \between Y \Rightarrow X \between (X \cap Y)$$

$$X \between \left(\bigcup_i Y_i\right) \Rightarrow \exists i (X \between Y_i)$$

(density)

$$\forall Z(X \between Z \;\Rightarrow\; Y \between Z) \quad \Rightarrow \quad X \subseteq Y$$

Overlap algebras (o-algebras)

Overlap algebra$(X, \leq, \gtrless)$ = complete lattice $(X, \leq)$ + binary relation $\gtrless$ s.t.

$$x \gtrless y \Rightarrow y \gtrless x$$

$$(x \gtrless y) \,\&\, (y \leq z) \Rightarrow (x \gtrless z)$$

$$x \gtrless y \Rightarrow x \gtrless (x \wedge y)$$

$$x \gtrless (\bigvee_{i \in I} y_i) \Rightarrow (\exists i \in I)\ (x \gtrless y_i)$$

$$\forall z\ (x \gtrless z \Rightarrow y \gtrless z) \Rightarrow x \leq y \qquad \text{(density)}$$

O-algebras vs cBa's

Classically,

overlap algebra = complete Boolean algebra

where $x \between y$ is $x \wedge y \neq 0$.

O-algebras vs cBa's

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Intuitionistically:
o-algebra $\not\Rightarrow$ cBa
cBa $\not\Rightarrow$ o-algebra

O-algebras are overt locales

Every o-algebra is a locale (frame)

$$x \wedge \bigvee_i y_i \;=\; \bigvee_i (x \wedge y_i)$$

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Every o-algebra is an overt (open) locale

$$\mathrm{Pos}(x) \Rightarrow \varphi \quad \text{iff} \quad x \leq \bigvee \{y \mid \varphi\}$$

where $\mathrm{Pos}(x)$ is $x \between x$.

An alternative (short) definition

Overlap algebra = overt locale X s.t.

$$(\forall z \in X)\,(\mathrm{Pos}(z \wedge x) \Rightarrow \mathrm{Pos}(z \wedge y)) \quad \Rightarrow \quad x \leq y$$

for all $x, y \in X$.

Examples?

Examples?

Given a cHa X, its stable elements form a cBa

$$X_{\neg\neg} \;=\; \{x \in X \,|\, x = \neg\neg x\}.$$

From the point of view of locale theory, $X_{\neg\neg}$ is the smallest dense sublocale of X.

(Isbell, Atomless parts of spaces, 1973)

strong density

A sublocale $X_j \hookrightarrow X$ is strongly dense if

$$j\left(\bigvee\{x \in X \mid \varphi\}\right) = \bigvee\{x \in X \mid \varphi\} \text{ for all } \varphi.$$

(Johnstone, A constructive ”closed subgroup theorem”..., 1989)

The smallest strongly dense sublocale of an overt locale

For X overt,
its smallest strongly dense sublocale
corresponds to the nucleus

$$r(y) = \bigvee \{x \mid \forall z. \mathrm{Pos}(x \wedge z) \Rightarrow \mathrm{Pos}(y \wedge z)\}$$

Classically: $r(y) = \neg\neg y$.

(C., Overlap algebras as almost discrete locales, arXiv:1601.04830)

The spatial case

For (X, τ) a topological space and for $V \in \tau$,

$$
\begin{aligned}
r(V) &= \bigcup\{U \in \tau \mid (\forall W \in \tau)\big((U \between W) \Rightarrow (V \between W)\big)\} \\
&= \bigcup\{U \in \tau \mid U \subseteq \operatorname{cl} V)\} \\
&= \operatorname{int} \operatorname{cl} V
\end{aligned}
$$

The smallest strongly dense sublocale of X is the overlap algebra of regular opens.

(C. & Sambin, The overlap algebra of regular opens, JPAA, 2010))

The smallest strongly dense sublocale of an overt locale is an overlap algebra,
and all overlap algebras arise in this way.

Two categories of overlap-algebras

o-Loc = o-algebras (seen as overt locales) +
open localic maps

OA = a suitable extension of **Rel**,
a suitable subcategory of **SupLat**

(Sambin)

The powerset functor

$$\mathcal{P} \quad : \quad \mathbf{Rel} \hookrightarrow \mathbf{SupLat}$$

$$X \xrightarrow{R} Y \qquad \mathcal{P}(X) \xrightarrow{\mathcal{P}(R)} \mathcal{P}(Y) \qquad Z \mapsto \{y \mid xRy \text{ for some } x \in Z\}$$

...between order-enriched categories:

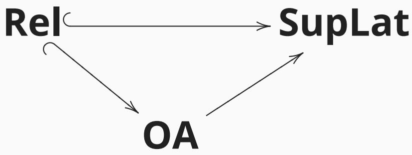
$R_1 \subseteq R_2$ iff $\mathcal{P}(R_1)(Z) \subseteq \mathcal{P}(R_2)(Z)$ for all Z

self-dualities

$$\mathbf{Rel} \longrightarrow \mathbf{Rel}^{op} \qquad\qquad \mathbf{SupLat} \longrightarrow \mathbf{SupLat}^{op}$$

$$\begin{array}{cccc} X & X & (X,\leq) & (X,\geq) \\ \downarrow R & \uparrow R^{\circ} & \downarrow f & \uparrow f_* \\ Y & Y & (Y,\leq) & (Y,\geq) \end{array}$$

$$yR^{\circ}x \Leftrightarrow xRy \qquad\qquad f \dashv f_*$$



We want to design **OA** so that it shows the same kind of self-duality as **Rel**.

characterising $(\_)^\circ$ via $\between$

$$yR^\circ x \Leftrightarrow xRy$$

$$\{x\} \between \mathcal{P}(R^\circ)(\{y\}) \Leftrightarrow \mathcal{P}(R)(\{x\}) \between \{y\}$$

$$Z \between \mathcal{P}(R^\circ)(Z') \Leftrightarrow \mathcal{P}(R)(Z) \between Z'$$

Symmetric/conjugate functions (o-relations)

$$
\begin{array}{ccc}
(X, \leq, \between) & \qquad & (X, \leq, \between) \\
\big\downarrow f & & \big\uparrow f^{\circ} \\
(Y, \leq, \between) & \qquad & (Y, \leq, \between)
\end{array}
$$

$$x \between f^{\circ}(y) \quad \text{iff} \quad f(x) \between y$$

Symmetric/conjugate functions (o-relations)

$$
\begin{array}{c}
(X, \leq, \gtrless) \\
\Big\downarrow f \\
(Y, \leq, \gtrless)
\end{array}
\qquad\qquad
\begin{array}{c}
(X, \leq, \gtrless) \\
\Big\uparrow f^{\circ} \\
(Y, \leq, \gtrless)
\end{array}
$$

$$x \gtrless f^{\circ}(y) \quad \text{iff} \quad f(x) \gtrless y$$

Classically:

$$x \wedge f^{\circ}(y) = 0 \quad \text{iff} \quad f(x) \wedge y = 0$$

(Jónsson & Tarski, Boolean algebras with operators, 1951)

Classically,
f admits a symmetric iff f preserves joins, and

$$f^{\circ}(y) = -f_{*}(-y)$$

The category **OA** of o-algebras and o-relations

$$\mathbf{OA}(X, Y) \; = \; \{f \in \mathbf{SupLat}(X, Y) \quad | \quad f^\circ \textit{ exists}\}$$

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$\mathcal{P} : \mathbf{Rel} \to \mathbf{OA}$ fully faithful and
$\mathcal{P}(R^\circ) = \mathcal{P}(R)^\circ$
(embedding between **dagger** categories)

Basic properties

$$f^{\circ\circ} = f$$

$$(gf)^\circ = f^\circ g^\circ$$

$$f \leq g \text{ iff } f^\circ \leq g^\circ$$

$$(\bigvee_i f_i)^\circ = \bigvee_i f_i^\circ$$ (**OA** is **SupLat**-enriched)

$$(f \cap g)^\circ = f^\circ \cap g^\circ$$ $(f \cap g \neq f \wedge g)$

(work in progress)

Rel is an allegory (in the sense of Freyd & Scedrov 1990)

What about **OA**?

Apparently, **OA** is not an allegory
(apparently, the modular law $gf \cap h \leq g(f \cap g^\circ h)$
only works for atomic algebras).

Nevertheless, **OA** satisfies many of the properties of a division allegory...

example

$f \leq ff^{\circ}f$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \between fx}{y \between f(1 \wedge x)}}{y \between f1 \wedge fx}\; f(1\wedge x) \leq f1 \wedge fx}{f1 \between y \wedge fx}}{1 \between f^{\circ}(y \wedge fx)}}{1 \between f^{\circ}y \wedge f^{\circ}fx}\; f^{\circ}(y \wedge fx) \leq f^{\circ}y \wedge f^{\circ}fx}{f^{\circ}y \between f^{\circ}fx}}{y \between ff^{\circ}fx}$$

(by density)

Maps in an order-enriched category

$f : X \to Y$ is a map if it has a right adjoint g

$$id_X \leq gf \quad \text{and} \quad fg \leq id_Y$$

Example: a map in **Rel** is a function (= total, single-valued relation), and

$$Map(\mathbf{Rel}) = \mathbf{Set}$$

Maps in **OA**

TFAE

i) f is a map in **OA**

ii) $f \dashv f^\circ$ (i.e. $f_* = f^\circ$)

iii) f° preserves finite meets
(and hence it is a frame homomorphism)

iv) f° corresponds to an open map of locales
(and $f = \exists_{f^\circ}$)

(C. & Contente, Overlap Algebras:..., LMCS, 2020)

$Map(\mathbf{OA})$ = “overlap” locales

= o-algebras + open localic maps

$Map(\mathbf{cBa}^{\bigvee}) = \mathbf{cBa}^{op}$ = Boolean locales

o-Loc = Map(**OA**)

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