Decision algorithm for a fragments of real analysis involving differentiable functions with convexity and concavity predicates on (semi-) open intervals

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July 3, 2020

Extended abstract

Around 1930, Alfred Tarski put forward an axiomatic system of elementary geometry based on first-order predicate calculus and proved, through a quantifier-elimination method, the completeness of his axiomatic theory. Completeness entailed the algorithmic solvability of the truth problem as referred to sentences in the elementary algebra of real numbers [5, 6]; subsequent improvements of the decision algorithm aimed, on the one hand, at making its complexity affordable in special subcases, on the other hand, at broadening its applicability beyond algebra, so as to handle entities and constructs relevant to the realm of real analysis. One of the decidable extensions was the *RMFC* theory (Theory of Reals with Monotone and Convex Functions), first investigated by D. Cantone, A. Ferro, E.G. Omodeo and J.T. Schwartz [3], whose language extends the existential theory of real numbers with various predicates about real functions. Other decidability results were achieved through ideas close to the ones which had led to the decidability of *RMFC*; those results regard: $RMFC^+$ (Augmented theory of Reals with Monotone and Convex Functions) by D. Cantone, G. Cincotti and G. Gallo [2], *RDF* (Theory of Reals with Differentiable Functions) by D. Cantone and G. Cincotti [1], and finally the RDF^+ theory (Augmented Theory of Reals with Differentiable Functions), the subject of this paper.

Let us briefly explain the basic idea through which one gets the decidability of *RMFC*, $RMFC^+$, *RDF*, and RDF^+. As a preliminary, notice that if $\psi = \exists x_1 \ldots \exists x_n \theta$ is an existential sentence of EAR, the elementary algebra of real numbers, and hence θ is a quantifier-free formula all of whose free variables are among $x_1, \ldots x_n$, then ψ is *true* iff there exist real numbers $a_1, \ldots, a_n$ which satisfy θ, to wit, iff θ is *satisfiable*; and, since by Tarski's result we have a decision algorithm for existential sentences of EAR, we also have a satisfiability algorithm for quantifier-free formulas of EAR. All four theories mentioned above extend the existential fragment of EAR with a new sort of variables, for functions, and with new relation symbols, for special relations between functions; in order to establish their decidability, we translate their formulas into logically equivalent formulas of the existential theory of real numbers and thus can rely upon Tarski's original decision result.

References

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