COMPUTING THE FOURIER DIMENSION

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An important notion in geometric measure theory and the theory of fractal dimension is Fourier dimension [3]. It estimates the "size" of a set by means of the Fourier transform of a probability measure supported on the set. More precisely, for every $A \subset \mathbb{R}^d$, let $\mathbb{P}(A)$ be the set of probability measures supported on A. For every $\mu \in \mathbb{P}(A)$, the Fourier transform $\widehat{\mu}$ of μ is defined as

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \, d\mu(x).$$

This allows us to define the Fourier dimension $\dim_{\mathrm{F}}(A)$ of A as the supremum of all $s \in [0, d]$ s.t.

$$(\exists c > 0)(\exists \mu \in \mathbb{P}(A))(\forall x \in \mathbb{R}^n)(|\widehat{\mu}(x)| \le c|x|^{-s/2}),$$

where $|\cdot|$ denotes the Euclidean norm. For a wide presentation on the Fourier dimension, see [1].

In this work we investigate the effective descriptive complexity of the Fourier dimension. We first work in the hyperspace $\mathbf{K}([0,1])$ of compact subsets of $[0,1]$, endowed with the Vietoris topology [2]. We show that, for every $p \in [0,1)$ (resp. $p \in (0,1]$), the family of closed subsets A of $[0,1]$ s.t. $\dim_{\mathrm{F}}(A) > p$ (resp. $\dim_{\mathrm{F}}(A) \ge p$) is Σ^0_2-complete (resp. Π^0_3-complete).

It is not trivial to generalize these results to higher dimensions: while the upper bounds rely on the compactness of the ambient space (and so the same proofs for the 1-dimensional case yield the upper bound when working in $[0,1]^d$), the hardness results are more delicate. Indeed, the notion of Fourier dimension is critically dependent on the ambient space. It is well-known that, if $n < d$, then every subset of a n-dimensional hyperplane has null Fourier dimension when seen as a subset of $[0,1]^d$. However, using some tools from harmonic analysis, we can prove that the complexities remain unchanged if we replace $[0,1]$ with $[0,1]^d$.

We then study the Fourier dimension for the family of closed subsets of $\mathbb{R}^d$, relaxing the compactness of the ambient space. Notice that, while there is a canonical choice for the topology on the hyperspace $\mathbf{K}([0,1]^d)$, this is not the case for the hyperspace $\mathbf{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$. We therefore consider both the Fell and the Vietoris topology for $\mathbf{F}(\mathbb{R}^d)$ (the two topologies coincide when working on a compact space).

In this case, while the hardness' results follow from the results for $[0,1]^d$, the upper bounds require extra care. We prove that, for a closed subset A of $\mathbb{R}^d$, the complexity of the properties $\dim_{\mathrm{F}}(A) > p$ and $\dim_{\mathrm{F}}(A) \ge p$ are again Σ^0_2-complete and Π^0_3-complete respectively, both if we endow $\mathbf{F}(\mathbb{R}^d)$ with the Fell or the Vietoris topology.

The Fourier dimension can be a useful tool to compute lower bounds for the Hausdorff dimension. Indeed, when working with Borel subsets of $\mathbb{R}^d$, it follows by Frostman's lemma that the Hausdorff dimension of a set A is the supremum of all

$s \in [0,d]$ s.t.

$$(\exists c > 0)(\exists \mu \in \mathbb{P}(A))(\forall x \in \mathbb{R}^n)(\forall r > 0)(\mu(B(x,r)) \leq cr^s),$$

where $B(x,r)$ denotes the ball of center x and radius r.

It can be shown that every probability measure that is supported on a Borel set and whose Fourier transform decays sufficiently quickly satisfies the above property and therefore $\dim_{\mathrm{F}}(A) \leq \dim_{\mathcal{H}}(A)$. The sets for which the two quantities are equal are called Salem sets.

We show that, for $d \geq 1$, every $p \in [0,d)$ (resp. $p \in (0,d]$) and every closed subset A of $[0,1]^d$ or $\mathbb{R}^d$ the property $\dim_{\mathcal{H}}(A) > p$ (resp. $\dim_{\mathcal{H}}(A) \geq p$) is Σ^0_2-complete (resp. Π^0_3-complete). Moreover, the results on the Hausdorff and the Fourier dimension yield the Π^0_3-completeness for the family of closed Salem subsets of $\mathbf{K}([0,1]^d)$ or $\mathbf{F}(\mathbb{R}^d)$.

Using the lightface completeness of the above properties, we can also characterize the Weihrauch degree of the maps that compute the Fourier dimension of a closed subset of $\mathbb{R}^d$ and that determine whether a closed set is Salem or not.

All the properties considered in this work are actually boldface complete for their respective class (the same proofs allow us to obtain both a boldface and a lightface completeness result).

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References

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