Internal Neighbourhood Structures

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1. Introduction

The idea of a *space* was conceived as an extra structure on a set and was initiated to explain the notions of *continuity* and *convergence*. The ensuing concept gave rise to prolific research yielding a vast body of knowledge. The idea behind this project was embedded in a quest to describe the effective descent morphisms of locales — presently conceived as a *point free* concept of *space*. It was observed that the Reiterman-Tholen method (see [6]) to embed ***Top*** inside an ambient locally cartesian closed topological hull was not available for ***Loc*** and necessitated internalising the notion of a *space* inside a suitable category. This extended abstract provides some of the progress and sketches the way forward.

2. Description of the Project

A finitely complete category $\mathbb{A}$ with finite coproducts, a proper (E, M)-factorisation system and for each object X of $\mathbb{A}$ the set $\boldsymbol{Sub}_{\mathsf{M}}(X)$ of M-subobjects (also called *admissible subobjects* of X) a complete lattice is called a *context*. Since every finitely complete, finitely cocomplete category with all intersections has a $(\boldsymbol{Epi}(\mathbb{A}), \boldsymbol{ExtMon}(\mathbb{A}))$-factorisation (or a $(\boldsymbol{ExtEpi}(\mathbb{A}), \boldsymbol{Mono}(\mathbb{A}))$-factorisation) structure, contexts abound in mathematics. Thus, in particular, the categories of ***Set***, ***FinSet***, ***Top***, ***Meas***, ***Loc***, $\boldsymbol{CRing}^{\mathrm{op}}$, …on one hand and ***Grp***, ***Ring***, ***CRing***, ***Frm***, ***SupLat***, K-***Mod*** (for any commutative unitary ring K), …, are all examples of a context.

Let $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context. For any object X, since $\boldsymbol{Sub}_{\mathsf{M}}(X)$ is a complete lattice, the set $\boldsymbol{Fil}(X)$ of all filters in $\boldsymbol{Sub}_{\mathsf{M}}(X)$ is complete compact algebraic lattice with $\uparrow p = \{x \in \boldsymbol{Sub}_{\mathsf{M}}(X) : p \leq x\}$ its compact elements. An order preserving map $\boldsymbol{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\mu} \boldsymbol{Fil}(X)$ is said to be *an internal preneighbourhood system on* X if $\mu \leq \uparrow$ (as order preserving maps), *internal weak neighbourhood system on* X if it is a interpolative preneighbourhood system, i.e., $x \in \mu(p) \Rightarrow (\exists y \in \mu(p))\big(x \in \mu(y)\big)$ and a *internal neighbourhood system on* X if it is meet preserving weak neighbourhood system. An internal neighbourhood system μ on X is an *internal topology on* X if the set $\mathfrak{O}\mu = \{x \in \boldsymbol{Sub}_{\mathsf{M}}(X) : x \in \mu(x)\}$ of μ-open subobjects of X is a frame in the complete lattice $\boldsymbol{Sub}_{\mathsf{M}}(X)$. A pair (X, μ) is said to be an *internal preneighbourhood space* or *internal weak neighbourhood space* or *internal neighbourhood space* or *internal topological space* if μ is an internal preneighbourhood or weak neighbourhood or neighbourhood system or topology, respectively on X.

Every morphism $X \xrightarrow{f} Y$ induces two adjunctions:

$$\boldsymbol{Sub}_{\mathsf{M}}(X) \underset{f^{-1}}{\overset{\exists_f}{\rightleftarrows}} \boldsymbol{Sub}_{\mathsf{M}}(Y) \quad \text{and} \quad \boldsymbol{Fil}(X) \underset{\overrightarrow{f}}{\overset{\overleftarrow{f}}{\leftrightarrows}} \boldsymbol{Fil}(Y) \tag{1}$$

and given the internal preneighbourhood spaces (X, μ), (Y, ϕ) a morphism $X \xrightarrow{f} Y$ is a *preneighbourhood morphism* if $\phi(y) \subseteq \overrightarrow{f}\mu(f^{-1}y)$ for each $y \in \boldsymbol{Sub}_{\mathsf{M}}(Y)$ or equivalently $\overleftarrow{f}\phi(\exists_f x) \subseteq \mu(x)$ for each $x \in \boldsymbol{Sub}_{\mathsf{M}}(X)$. If (X, μ) and (Y, ϕ) are internal neighbourhood spaces then a preneighbourhood morphism

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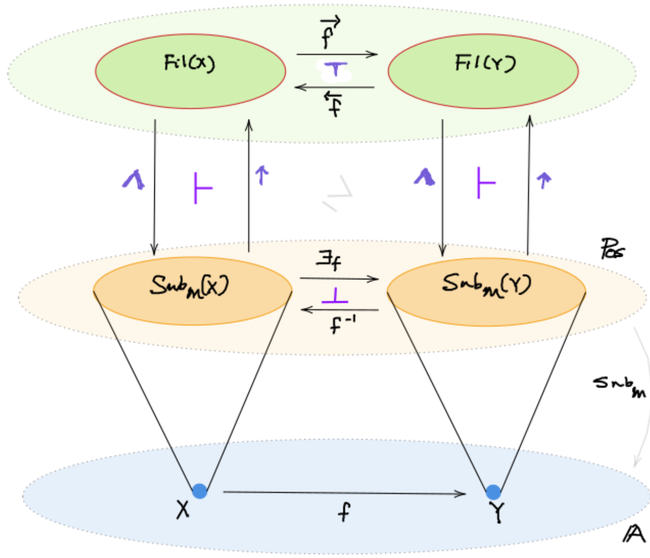

FIGURE 1. Schematic diagram of a *structure* in internalising *spaces*

$(X,\mu)\xrightarrow{f}(Y,\phi)$ for which f^{-1} preserve arbitrary joins is a *neighbourhood morphism*. Hence $\boldsymbol{pNbd}[\mathbb{A}]\supseteq_{\text{full}}$ $\boldsymbol{wNbd}[\mathbb{A}]\supseteq_{\text{non-full}}\boldsymbol{Nbd}[\mathbb{A}]\supseteq_{\text{full}}\boldsymbol{Top}[\mathbb{A}]$ are different categories of internal neighbourhood spaces.

The structure described above can be described in terms of the entities in Figure 1, wherein it is observed that $\boldsymbol{Sub}_{\mathsf{M}}(-)$ is a fibration over $\mathbb{A}$ from $\boldsymbol{Pos}$. The aim of the project is to investigate the extent to which this dependence carries through.

3. List of Results

The following is a list of results obtained for a context $\mathcal{A}$, each of which only utilises the diagram in Figure 1. A context $\mathcal{A}$ is said to be *extensive* if $\mathbb{A}$ is extensive (see [1]).

[4, Theorem 3.43]: $\boldsymbol{wNbd}[\mathbb{A}]$ is a bireflective full subcategory of $\boldsymbol{pNbd}[\mathbb{A}]$.

[4, Theorem 3.44]: Let $\boldsymbol{wNbd}[\mathbb{A}]_{\text{ppj}}$ be the subcategory of $\boldsymbol{wNbd}[\mathbb{A}]$ consisting only of preneighbourhood morphisms f for which f^{-1} preserve arbitrary joins. Then $\boldsymbol{Nbd}[\mathbb{A}]$ is a full bireflective subcategory of $\boldsymbol{wNbd}[\mathbb{A}]_{\text{ppj}}$.

[4, Theorem 4.7]: $\boldsymbol{Top}[\mathbb{A}]$ is a full bireflective subcategory of $\boldsymbol{Nbd}[\mathbb{A}]$ if and only if $\boldsymbol{Top}[\mathbb{A}]$ is topological over $\mathbb{A}_{ppj}$ if and only if for every object X there exists a largest internal topology on X, where $\mathbb{A}_{ppj}$ is the subcategory of $\mathbb{A}$ consisting of morphisms f for which f^{-1} preserve arbitrary joins. Every admissible monomorphism is in $\mathbb{A}_{ppj}$ if and only if each $\boldsymbol{Sub}_{\mathsf{M}}(X)$ is a frame.

[4, Theorem 4.8]: Both $\boldsymbol{pNbd}[\mathbb{A}]$ and $\boldsymbol{wNbd}[\mathbb{A}]$ are topological over $\mathbb{A}$ while $\boldsymbol{Nbd}[\mathbb{A}]$ is topological over $\mathbb{A}_{ppj}$.

[5, Theorem 4.5, 4.9 & 4.13]: Let (X,μ) be an internal preneighbourhood space. Define:

$$\text{cl}_\mu p=\bigvee\{x\in\boldsymbol{Sub}_{\mathsf{M}}(X): x\neq\mathbf{1}_X \text{ and } u\in\mu(x)\Rightarrow u\wedge p\neq\mathbf{0}_X\},\qquad \text{for } p\in\boldsymbol{Sub}_{\mathsf{M}}(X). \tag{2}$$

Assume all morphisms *reflect zero*, i.e., for every morphism $X\xrightarrow{f}Y$, $f^{-1}\mathbf{0}_Y=\mathbf{0}_X$. Then, cl_μ defines a grounded transitive categorical closure operator on $\boldsymbol{pNbd}[\mathbb{A}]$ (see [2] for definitions). Furthermore, the closure operator is finitely additive if every filter on X extends to a prime filter (see [3] for details of filters extending to prime filters in a general poset).

[5, Theorem 4.23]: A morphism $X\xrightarrow{f}Y$ from an internal preneighbourhood space (X,μ) to an internal preneighbourhood space (Y,ϕ) is said to be *closed* if it preserves closed subobjects. The set $\mathbb{A}_{\text{closed}}$ of all closed morphisms of $\mathbb{A}$ contain all isomorphisms, is closed under compositions, closed embeddings are pullback stable and if a composite $g_\circ f$ is closed then g is a closed morphism if f is a closed preneighbourhood morphism hereditarily in E.

[5, Theorem 5.6]: In an admissible context $\mathcal{A}$, the admissible monomorphisms are closed under finite sums if and only if the monomorphisms in E between finite sums are stable under pullbacks along coproduct injections.

[5, Theorem 5.7, Corollary 5.10]: In an admissible context $\mathcal{A}$, the sum of closed subobjects is a closed subobject if and only if the coproduct injections are closed and finite sums of closed embeddings is in M.

In particular, sum of closed morphisms is closed.

[5, Theorem 5.12]: In an admissible context $\mathcal{A}$ let $K(X)$ denote either $\boldsymbol{Sub}_{\mathsf{M}}(X)$ or the set of closed subobjects of X, where (X, μ) is an internal preneighbourhood space. Under assumptions of closure as made above, $K(X + Y)$ is the biproduct of $K(X)$ and $K(Y)$.

4. Goals

Keeping in view the diagram in Figure 1 the following are some of the topics that are being investigated:

(a) the structure of proper morphisms
(b) the separated objects and hence the ensuing category of Hausdorff objects
(c) the compact objects and the category of compact Hausdorff objects
(d) the Hausdorff and compact Hausdorff reflections
(e) investigation of locally cartesian closed extensions of $\boldsymbol{pNbd}[\mathbb{A}]$, $\boldsymbol{wNbd}[\mathbb{A}]$ and $\boldsymbol{Nbd}[\mathbb{A}]$ and hence the enquiry into the structure of effective descent morphisms of the categories of internal neighbourhood spaces
(f) investigation into semi-abelian internal preneighbourhood spaces and their associated homologies
(g) extension of the diagram in Figure 1 to a general fibration with suitable properties to ensure the conclusions reached above.

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