KRUSKAL'S TREE THEOREM AND CONTINUOUS TRANSFORMATIONS OF PARTIAL ORDERS

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We present recent work about a uniform version of Kruskal's theorem, as well as a new result about a restricted version of the latter. Our work makes crucial use of continuous transformations of partial orders, and of computable constructions relative to these transformations.

A partial order $X = (X, \leq_X)$ is a well partial order (wpo) if the following holds: for any infinite sequence $x_0, x_1, \ldots \subseteq X$, there are indices $i < j$ with $x_i \leq_X x_j$. The usual Kruskal theorem [6] asserts that the set of finite trees, ordered by embeddability, is a well partial order. This has important implications for theoretical computer science, in particular in the context of term rewriting.

In our uniform Kruskal theorem, we replace the collection of finite trees by general recursive data types. To make this precise, we will first recall the recursive construction of finite trees. By using notions from category theory, we prepare the desired generalization.

Let us write $W(X)$ for the set of finite sequences with entries in X. Given $f : X \to Y$, we define a function $W(f) : W(X) \to W(Y)$ by setting $W(f)(\langle x_0, \ldots, x_{n-1}\rangle) := \langle f(x_0), \ldots, f(x_{n-1})\rangle$. Up to isomorphism, there is a unique pair of an order $\mathcal{T}W$ and a bijection

$$\kappa : W(\mathcal{T}W) \xrightarrow{\cong} \mathcal{T}W$$

that is initial in the following sense: for any function $\pi : W(X) \to X$, there is a unique function $f : \mathcal{T}W \to X$ with $f \circ \kappa = \pi \circ W(f)$. Indeed, we can identify $\mathcal{T}W$ with the set of finite (structured) trees: the element $\kappa(\langle t_0, \ldots, t_{n-1}\rangle) \in \mathcal{T}W$ corresponds to the tree in which the root has immediate subtrees $t_0, \ldots, t_{n-1} \subseteq \mathcal{T}W$.

The usual notion of tree embedding can be reconstructed in the same spirit: if $X = (X, \leq_X)$ is a partial order, we define a partial order $\leq_{W(X)}$ on $W(X)$ by stipulating that

$$\langle x_0, \ldots, x_{m-1}\rangle \leq_{W(X)} \langle x'_0, \ldots, x'_{n-1}\rangle$$

holds if, and only if, there is a strictly increasing $f : \{0, \ldots, m-1\} \to \{0, \ldots, n-1\}$ with $x_i \leq_X x'_{f(i)}$ for all $i < n$ (which is the order from Higman's lemma). Writing $[X]^{<\omega}$ for the set of finite subsets of X, we define $\operatorname{supp}_X : W(X) \to [X]^{<\omega}$ by

$$\operatorname{supp}_X(\langle x_0, \ldots, x_{n-1}\rangle) := \{x_0, \ldots, x_{n-1}\}.$$

We can recursively define a partial order $\leq_{\mathcal{T}W}$ on $\mathcal{T}W$ by stipulating

$$(\star) \qquad \kappa(\sigma) \leq_{\mathcal{T}W} \kappa(\tau) \quad \Leftrightarrow \quad \sigma \leq_{W(\mathcal{T}W)} \tau \text{ or } \kappa(\sigma) \leq_{\mathcal{T}W} t \text{ for some } t \in \operatorname{supp}_{\mathcal{T}W}(\tau).$$

This coincides with the usual notion of tree embedding: the inequality $\sigma \leq_{W(\mathcal{T}W)} \tau$ corresponds to the case where immediate subtrees of $\kappa(\sigma)$ are mapped into subtrees of $\kappa(\tau)$, while the root is mapped to the root; for $\kappa(\sigma) \leq_{\mathcal{T}W} t$, the entire tree $\kappa(\sigma)$ is mapped into the subtree t of $\kappa(\tau)$.

In our uniform Kruskal theorem, we allow W to range over a large class of transformations of partial orders. To ensure the existence of fixed points (and to keep the meta theory weak), we require that W is continuous in a suitable sense. Let us write PO for the category of partial orders and quasi embeddings (i. e., functions $f : X \to Y$ such that $f(x) \leq_Y f(x')$ implies $x \leq_X x'$). The following is analogous to Girard's dilators on linear orders [4].

Definition 1. A PO-dilator consists of a functor $W : \mathrm{PO} \to \mathrm{PO}$ and a natural family of functions $\operatorname{supp}_X : W(X) \to [X]^{<\omega}$ for which the following "support condition" holds: if $f : X \to Y$ is an embedding (not just a quasi embedding), then $W(f) : W(X) \to W(Y)$ is an embedding with range

$$\operatorname{rng}(W(f)) = \{\sigma \in W(Y) \mid \operatorname{supp}_Y(\sigma) \subseteq \operatorname{rng}(f)\}.$$

A PO-dilator $W = (W, \mathrm{supp})$ is called normal if we have

$$\sigma \leq_{W(X)} \tau \quad \Rightarrow \quad \text{for any } x \in \mathrm{supp}_X(\sigma) \text{ there is an } x' \in \mathrm{supp}_X(\tau) \text{ with } x \leq_X x',$$

for any partial order X and all elements $\sigma, \tau \in W(X)$. Finally, we call W a WPO-dilator if $W(X)$ is a well partial order whenever the same holds for X.

The support condition ensures that PO-dilators preserve direct limits and pullbacks. Due to this continuity property, they can be represented by subsets of $\mathbb{N}$ (similarly to higher type functionals over the natural numbers). Relative to the representation of a normal PO-dilator W, one can compute a set $\mathcal{T}W$ and a bijection $\kappa : W(\mathcal{T}W) \to \mathcal{T}W$. By $(\star)$ we get a partial order on $\mathcal{T}W$ (cf. the construction by Hasegawa [5]). The resulting objects are initial in a suitable sense. While the construction of $\mathcal{T}W$ can be implemented in the usual base theory $\mathbf{RCA}_0$ of reverse mathematics, we obtain a very strong statement if we consider the preservation of well partial orders:

Theorem 2 ([3]). *The following are equivalent over* $\mathbf{RCA}_0$ *with the chain-antichain principle:*

(i) if W is a normal WPO*-dilator, then $\mathcal{T}W$ is a well partial order,*

(ii) the principle of Π^1_1-comprehension holds.

We refer to statement (i) as the uniform Kruskal theorem. By the above considerations, the usual Kruskal theorem is an instance of it. Higman's lemma is another instance, as $W(X) := 1 + Z \times X$ generates sequences with entries in Z. We point out that the proof of Theorem 2 makes crucial use of a result from ordinal analysis [1, 2].

In our opinion, the equivalence in Theorem 2 is particularly interesting for the following reason: a very elegant proof of Kruskal's theorem uses Nash-Williams's minimal bad sequence method [8]. The latter is equivalent to Π^1_1-comprehension (due to Marcone [7]), and hence much stronger than the usual Kruskal theorem. Theorem 2 shows that our uniform Kruskal theorem exhausts the full power of minimal bad sequences.

We conclude with a new result on a restricted version of the uniform Kruskal theorem. Say that a PO-dilator $W = (W, \mathrm{supp})$ has bounded arity if there is an $n \in \mathbb{N}$ with the following property: for any partial order X and any $\sigma \in W(X)$, the finite set $\mathrm{supp}_X(\sigma) \subseteq X$ has at most n elements. Intuitively speaking, this bounds the arity of constructor symbols: the set of binary trees can be constructed as the least fixed point $\mathcal{T}W$ of the transformation $X \mapsto W(X) := 1 + X^2$ of bounded arity (with $n = 2$); on the other hand, we need constructors of all arities to generate all finite trees.

Theorem 3 (F., Rathjen, Weiermann 2020). *The uniform Kruskal theorem for normal* WPO*-dilators of bounded arity follows from the usual Kruskal theorem for labelled trees, over* $\mathbf{ACA}_0$.

In view of Theorem 2, the uniform Kruskal theorem (for unbounded arity) cannot follow from a Π^1_2-statement such as the usual Kruskal theorem with labels. Theorem 3 shows that the logical complexity drops for bounded arity. At first, this is quite surprising; however, there is a corresponding phenomenon for dilators on linear orders, which was discovered by Girard [4].

References

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