Descriptive complexity on non-Polish spaces

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In order to compute on general topological spaces, computable analysis and its Type-2 Theory of Effectivity (TTE) suggest to use representations to encode the elements of the space by objects that can be operated on by computers, for example infinite sequences of bytes [5]. With the definition of represented spaces, we obtain a general context in which computability has been extended.

Descriptive Set Theory (DST) and its effective version provide us with a natural framework to study the complexity of sets (in a represented space) according to two points of view. On the one hand, the *topological* complexity of a Borel set S is a description of its structure according to the way it is built from open sets. On the other hand, its *algorithmic* complexity is the complexity of its encoding by the chosen representation, and can be viewed as the complexity of deciding wheter a point belongs to S with an algorithm working on infinite sequences of bits and allowed to change its mind.

While DST first focused on Polish spaces (complete metrizable spaces with a dense countable subset), it was later extended to include ω-continuous domains [4] and more recently quasi-Polish spaces [1]. In this presentation, we ask the following question on more general classes of spaces: are these two approaches (topological and algorithmic) always equivalent? More formally, we investigate these two questions:

1. Is it equivalent for a set to be of topological complexity Γ and of algorithmic complexity Γ ?
2. How to establish lower-bounds on topological and algorithmic complexities of sets? How relevant is hardness?

While it is long known that topological and algorithmic complexities are always equivalent in the context of Polish spaces, in [1] this equivalence was extended on the larger class of countably-based spaces. In this talk, we provide two important results for the compatibility of TTE and DST on countably-based spaces [2]. The first one effectivizes the equivalence of algorithmic and topological complexities with effective complexity classes, which in our opinion illustrates that countably-based spaces are an ideal framework for computability and TTE. Secondly, we develop a criterion based on the notion of hardness in order to prove lower-bounds on topological complexities.

We also study CoPolish spaces as a class of spaces that do not have a countable base of open sets. In particular, we focus on the space of real polynomials equipped with the CoPolish topology, and exhibit a counter-example showing a mismatch between our two algorithmic and topological complexities [2]. While a characterization of the spaces on which these two notions of complexity are equivalent has yet to be found, we suggest this divergence is related to the mismatch between topological and sequential aspects of the topology. Indeed, we demonstrate that a CoPolish space is Fréchet-Urysohn (topological closures are the same as sequential closures) if and only if topological and algorithmic complexities coincide on their first levels. We also prove that the usual notion of hardness reflects algorithmic complexity, and we introduce a notion of hardness that captures topological complexity [3].

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