Using differential equations to characterize complexity classes

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Classical complexity theory deals with computation over discrete entities. However, several connections between classical complexity theory and problems from Analysis are known, e.g. such as those presented in [4], [3], [2], [1]. From the computability point of view, in [1] it has been shown that every computable function can be successfully simulated as a solution of certain polynomial ordinary differential equations (ODE). From the complexity point of view, instead, in [2] the authors showed how the class of real functions computable (in the sense of Computable Analysis) in polynomial time can be characterized in terms of a class of dynamical systems defined with polynomial ODEs using polynomial bounds. From this result it was also established in [2] a characterization of P using ODEs. These two results combined suggested that a similar characterization could be theoretically achieved also for higher complexity classes such as $EXPTIME$, the Grzegorczyk hierarchy, etc. Nevertheless, the latter is not immediate as some properties which are used in the proof of the result of [2] and which are true for polynomials, such as closure under composition, no longer hold e.g. for exponential functions. Therefore, some questions arise. What modifications should be forced into the structure of the polynomial result to ensure flexibility for higher classes as well? Is it enough to substitute polynomials by appropriate functions (exponentials, etc.) in the definition presented in [2]?

Here we use consider the class of functions computable in exponential time as a toy example to understand what must happen so that we can present characterizations of other complexity classes in terms of bounded ODEs in a similar spirit to what was done in [2].

We next define Analog Time-Space Polynomial (or ATSP) functions as in [2]. In what follows, $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{K}$ is a fixed, predefined field.

Definition 1 *Let $f :\subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then $f \in ATSP$ if and only if there exist $d \in \mathbb{N}$, and $p \in \mathbb{K}^d[\mathbb{R}^d]$, $q \in \mathbb{K}^d[\mathbb{R}^n]$ and polynomials $\Pi : \mathbb{R}^2{}_+ \to \mathbb{R}_+$ and $\Upsilon : \mathbb{R}^2{}_+ \to \mathbb{R}_+$ such that for any $x \in domf$, there exists (a unique) $y : \mathbb{R}_+ \to \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}^+$:*

- $y(0) = q(x)$ *and* $y'(t) = p(y(t))$
- $\forall \mu \in \mathbb{R}_+$ *if* $t \geq \Pi(||x||, \mu)$ *then* $||y_{1..m}(t) - f(x)|| \leq e^{-\mu}$
- $||y(t)|| \leq \Upsilon(||x||, t)$

In [2] it is shown that, with a proper definition of an equivalence relation, the class of Analog Time-Space Polynomial functions is equivalent to the one of discrete functions computable in polynomial time, FP.

We note that the solution of the system is always bounded by a polynomial $\Upsilon(||x||, t)$; moreover if the time of the system t is greater that $\Pi(\|x\|, \mu)$ then the ODE approaches the value of $f(x)$ with accuracy bounded by $e^{-\mu}$.

In the present work we define the class of Analog Time-Space Exponential functions as in Definition 1, with the exception that we allow Π and Υ to depend exponentially on $\|x\|$ but only *polynomially* on μ (and this is one of the crucial novel aspects made clear from the present work). Then we get the following counterpart of [2].

Theorem 2 *The class of Analog Time-Space Exponential functions is equivalent to the class of descrete functions which are computable in exponential time, FEXPTIME.*

As a corollary, this result can be used to give an analog characterization of EXPTIME. Using the insight obtained that the dependence on μ should be polynomial even for non-polynomial complexity classes (at least those which are super-polynomial), we can show that a similar characterization can be made for the whole Grzegorczyk hierarchy. This abstract describes work in progress.

References

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