On the logical and computational properties of the uncountable

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I provide an overview of my joint project with Dag Normann on the Reverse Mathematics and computability theory of the uncountable ([5–9]), with an emphasis on the latter. We show that the following basic theorems are *hard to prove* relative to the usual 'normal' scale of (higher-order) comprehension axioms, while the objects claimed to exist by these theorems are similarly *hard to compute*, in the sense of Kleene's computational framework given by his S1-S9 schemes ([4]). In each case, full second-order arithmetic comes to the fore, while the *Axiom of Choice* is not needed.

i. There is no injection (or bijection) from $[0, 1]$ to $\mathbb{N}$ (Cantor, [2]).
ii. Arzelà's convergence theorem for the *Riemann* integral ([1]).
iii. Item ii with Tao's *metastability* instead of *convergence* in the conclusion.
iv. Baire category theorem for open sets as characteristic functions.
v. Covering lemmas for *uncountable* covers of the unit interval.
vi. Covering lemmas for *countable* covers of the unit interval, using the usual definition of 'countable set'.
vii. Bolzano-Weierstrass: a countable set in $[0, 1]$ has a supremum, using the usual definition of 'countable set'.
viii. *König's lemma* and *Ramsey's theorem* in their original formulation ([3, 10]).

We discuss how the 'normal' scale, based on comprehension and discontinuous functionals, is unsuitable as a measure of logical and computational strength in this context. We introduce an alternative 'non-normal' scaled, based on a (classically valid) continuity axiom from Brouwer's intuitionistic mathematics, called the *neighbourhood function principle* ([11]).

As it turns out, item i. above is the weakest principle on the non-normal scale, while computing realisers still requires full second-order arithmetic (in terms of comprehension/discontinuous functionals).

We discuss an annoying open conjecture related to the above items, as follows. For item v there should be a difference between computing finite sub-covers (as in the Heine-Borel theorem) and computing an upper bound on the size of finite sub-covers; we do not have any idea how to exhibit this difference. Similar conjectures exist for e.g. item i.

Finally, this study is motivated by recent results ([9]) showing that the coding practise common in most of computability theory significantly change the logical and computational strength of basic theorems *about the Riemann integral*.

References

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