Extending the extensional level of the Minimalist Foundation to axiomatic set theories

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In Martin-Löf type theory **MLTT** propositions are interpreted as types, according to the usual correspondence between connectives and quantifiers and type constructors. As a consequence, the Lindenbaum-Tarski doctrine of **MLTT** satisfies the Choice Rule. Indeed, if $\mathbf{LT_{MLTT}} \vdash \forall x \in A \exists y \in B\, R(x, y)$, then there exists a term $t(x) \in B\,[x \in A]$ such that $\mathbf{LT_{MLTT}} \vdash \forall x \in A\, R(x, t(x))$. The same holds for every extension of **MLTT**, provided that we still adopt the propositions-as-types paradigm. At the same time, the choice rule does not hold in the Lindenbaum-Tarski hyperdoctrines defined on the categories of definable classes and definable functions of set theories **CZF**, **IZF**, **ZF** and **ZFC**. The reason is that if choice rule holds for such an hyperdoctrine, then existence property **EP** in [2] would hold for the relative theory; but it is known that **EP** is not satisfied by **CZF**, **IZF**, **ZF** and **ZFC** (if they are consistent). This in particular means that the theory **MLTT** cannot be extended to a theory equivalent to one of these set theories, at least if we want to preserve logic.

In this talk we want to show that the extensional level of the Minimalist Foundation in [1] can be extended (by adding rules to the system) to theories which are equivalent to **CZF**, **IZF**, **ZF** and **ZFC**.

The Minimalist Foundation consists of two levels which should work as "motherboards" on which "expansion cards" can be installed to obtain the most relevant classical and intuitionistic, predicative and impredicative, foundational theories in the literature.

Both levels of the Minimalist Foundation are expressed in a type theorical language, one intensional and the other extensional. While intensional type theories like intensional Martin-Löf type theory or Coquand's Calculus of Constructions can be obtained by adding specific rules to the intensional level **mTT**, extensional theories like set theories should be obtained as extensions of the extensional one **emTT**.

We will show that this is possible if one follows these four steps:

1. we add rules which make natural to think of collections of **emTT** as definable classes of set theory;

2. we add rules which identify sets of **emTT** with definable sets of set theory;

3. we add rules and axioms of set theory to the system;

4. we add rules which describe an interpretation of types and terms of **emTT** in set theory.

Riferimenti bibliografici

[1] M. E. Maietti and G. Sambin. Toward a minimalist foundation for constructive mathematics. In L. Crosilla and P. Schuster, editor, *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*, number 48 in Oxford Logic Guides, pages 91–114. Oxford University Press, 2005.

[2] Michael Rathjen. The disjunction and related properties for constructive Zermelo-Fraenkel set theory. *J. Symbolic Logic*, 70(4):1232–1254, 2005.