On the order-enriched category of overlap-algebras

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Some years ago, while developing his constructive approach to Topology (both in the pointwise and in the pointfree sense), the so-called "Basic Picture" [6], Giovanni Sambin introduced the notion of an "overlap algebra" as a convenient constructive axiomatization of the structure of all subsets of a given set [3]. It turned out, then, that an overlap algebra can always be seen as the smallest strongly dense sublocale of an overt locale [1]: with classical logic, this is just a Boolean locale, that is, a complete Boolean algebra.

In Sambin's Basic Picture, relations between sets play a fundamental role (for instance, continuity is defined in terms of a commutative diagram of relations). So an abstract notion of a relation between overlap algebras was introduced which, assuming excluded middle, boils down to a join-preserving function between complete Boolean algebras.

Here we consider the order-enriched category of overlap algebras, where the morphisms are given by the notion of an abstract relation mentioned above, and its classical counterpart, namely the order-enriched category of complete Boolean algebras and join-preserving functions (in both cases, we consider the pointwise order on functions).

We show (constructively in the case of overlap algebras, and classically in the case of complete Boolean algebras) that these two categories are dagger categories, actually categories with involution, and extend the usual 2-category of relations. Moreover, they satisfy many of the properties of a unitary, division allegory.

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An allegory [4, 5] is a particular kind of order-enriched category which enjoys the typical properties of a category of relations over a regular category. A "map" in an allegory is a morphism with a right adjoint (in the case of the allegory of relations, a map is just a total single-valued relation, that is, a function). Relative to the additional properties an allegory can have, the subcategory of maps can even be a topos.

In our case, the corresponding category of "maps" is classically equivalent to the category of Boolean locales. Constructively, it is the category whose objects are overlap algebras (regarded as particular overt locales) and whose morphisms are open maps between them [2].

References

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