Interior operators generated by ideals in complete domains

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Due to the close connection between Galois connections and interior operators, in this article we study the properties of a special class of interior operators generated by ideals. The mathematical framework in which we work is given by complete domains, i.e. complete posets in which the set of minimal elements is a basis. In the first part we present properties in complete domains which are useful in the next constructions. Then we build a new type of interior operator denoted by $G(i, I)$, starting from a given interior operator i and an ideal I. Several results related to these interior operators are presented, as well as the links between the properties of the ideal I and the properties of the operator $G(i, I)$. Two important properties denoted by $\mathcal{P}_i$ and $\mathcal{Q}_i$ are analyzed and characterized. Various characterizations for compact elements (in a generalized sense) are given. In the last part, for an arbitrary interior operator i, we build ideals having the $\mathcal{P}_i$ property, and ideals having the $\mathcal{Q}_i$ property.

Let $(X, \leq)$ be a partially ordered set (poset), and $A \subseteq X$. We denote by $ub(A)$ the upper bounds set of A, and by $lb(A)$ the lower bounds set of A. For a cardinal number $\alpha \geq 2$, the set A is called α-directed if for all $B \subseteq A$ with $|B| \leq \alpha$, we have $A \cap ub(B) \neq \emptyset$. The set A is called α-ideal if it is a lower set and α-directed set; A is a complete ideal if it is a lower and α-directed set for any cardinal $\alpha \geq 2$. If α is a finite cardinal, then we call ideal any α-ideal. Let $\mathcal{I}_\alpha(X)$ be the set of α-ideals of $(X, \leq)$, and $\mathcal{I}(X)$ be the set of ideals of $(X, \leq)$. If $\mathcal{L} \subseteq \mathcal{P}(X)$, we denote by $\mathcal{L}_d$ the family of directed sets of $\mathcal{L}$. If $(X, \leq)$ is a pointed poset, an element $x \in X$ is called minimal if $x \neq \perp$ and $\downarrow x = \{x, \perp\}$. We denote by $min(X)$ (or shortly min) the set of minimal elements of $(X, \leq)$.

Let $(X, \leq)$ be a pointed poset, $\mathcal{L}$ be a non-empty family of subsets of X, and $g : X \to X$ a function. For any $x, z \in X$ we say that z approximates x regarding $(g, \mathcal{L})$ or z is an essential part of x regarding $(g, \mathcal{L})$ if for each set $A \in \mathcal{L}$ such that there exists $\sqcup A$ and $x \leq \sqcup A$, there is $a \in A$ such that $z \leq g(a)$. In this case we denote $z \ll_{(g,\mathcal{L})} x$. We say that x is $(g, \mathcal{L})$-compact if it approximates itself regarding $(g, \mathcal{L})$. For $\mathcal{L} = \mathcal{P}(X)$ and $g = 1_X$, $z \ll_{(g,\mathcal{L})} x$ is denoted by $z \ll x$; we say that z approximates x. Also, x is called compact if x is $(1_X, \mathcal{P}(X))$-compact. The set of $(g, \mathcal{L})$-compact elements is denoted by $K_{g,\mathcal{L}}(X)$, and the set of compact elements is denoted by $K(X)$. For any $x \in X$, we denote by $\twoheaddownarrow x$ the set $\{z \in X \setminus \{\perp\} \mid z \ll x\}$. A set $B \subseteq X$ is called a basis if for all $x \in X \setminus \{\perp\}$ we have $\sqcup(B \cap \twoheaddownarrow x) = x$. A function $f : X \to X$ is called $(g, \mathcal{L})$-Scott-continuous if f is isotone and for all $A \in \mathcal{L}$ such that there is $\sqcup A$, there exists $\sqcup f(A)$ and $g(\sqcup f(A)) = gf(\sqcup A)$. Also, f is called Scott-continuous if f is $(1_X, \mathcal{P}(X))$-Scott-continuous. If $\alpha \geq 2$ is a cardinal number and $A \subseteq X$, we denote by $S_\alpha(A) = \{\sqcup M \mid M \subseteq A, |M| \leq \alpha\}$ and $S(A) = \{\sqcup M \mid M \subseteq A,\ M \text{ finite}\}$.

Definition 1. *A poset $(X, \leq)$ is a complete domain if X is complete and min is a basis.*

Proposition 1. *Any complete domain is an algebraic domain.*

Let $(X, \leq)$ be a complete domain, and $c : X \to X$ be the function $c(x) = \sqcup\{y \in X \mid y \sqcap x = \perp\}$ for all $x \in X$. To simplify the notation, we use cx instead of $c(x)$.

Definition 2. *A function $i : X \to X$ is an interior operator on X if $i \leq 1_X$, $i^2 = i$, and $i(\sqcap A) = \sqcap i(A)$ for all finite subsets A of X.*

Let i be an interior operator on X, and $I \in \mathcal{I}(X)$. For each $x \in X$ we define the set $A_{i,I}(x) = \{m \in min \mid \exists u \in i(X) \text{ such that } m \leq u \text{ and } u \sqcap x \in I\}$. Let $g_{i,I} : X \to X$ be the function defined by $g_{i,I}(x) = \sqcup A_{i,I}(x)$ if $A_{i,I}(x) \neq \emptyset$, and $g_{i,I}(x) = \perp$ if $A_{i,I}(x) = \emptyset$.

Proposition 2. *Let $U \subseteq X$, $\widetilde{U} = \{u \sqcap cy \mid u \in U, y \in I\}$, $\mathcal{L} = \mathcal{P}(U)$, and $\widetilde{\mathcal{L}} = \mathcal{P}(\widetilde{U})$. Let $h : X \to X$ be a function. Then $K_{hg_{i,I},\mathcal{L}}(X) = K_{hg_{i,I},\widetilde{\mathcal{L}}}(X)$.*

If $x \in I$, then $x \leq g_{i,I}(x)$. If the converse is also true, we say that I has the property $\mathcal{P}_i$.
Thus, $\mathcal{P}_i = \{I \in \mathcal{I}(X) \mid x \leq g_{i,I}(x) \Rightarrow x \in I\}$.

Proposition 3. *$I \in \mathcal{P}_i$ if and only if $x \sqcap g_{i,I}(x) \in I$ for all $x \in X$.*

We say that I has the property $\mathcal{Q}_i$ if $g_{i,I}(\top) = \bot$. Thus, $\mathcal{Q}_i = \{I \in \mathcal{I}(X) \mid g_{i,I}(\top) = \bot\}$.
Then $I \in \mathcal{Q}_i$ iff $i(X) \cap I = \{\bot\}$. Moreover, if $I \in \mathcal{Q}_i$, then $g_{i,I} = ic$ on $i(X)$.

Theorem 1. *The function $G_{i,I} = 1_X \sqcap g_{i,I}c$ is an interior operator on X such that $i \leq G_{i,I}$.*

We also have $i(X) \subseteq G_{i,I}(X)$, $G_{i,I}i = iG_{i,I} = i$, $G_{i,I}(X) \subseteq \{u \sqcap c(v \sqcap g_{i,I}(v)) \mid u \in i(X), v \in X\}$, and $G_{i,I}(X) = \{u \sqcap c(v \sqcap g_{i,I}(v)) \mid u \in i(X), v \in X\}$ iff $g_{i,I}(v \sqcap g_{i,I}(v)) = \top$ for all $v \in X$. Therefore, if $I \in \mathcal{P}_i$, then $G_{i,I}(X) = \{u \sqcap cy \mid u \in i(X), y \in I\}$.

Theorem 2. *Let $\mathcal{L} = \mathcal{P}(i(X))$, $\widetilde{\mathcal{L}} = \mathcal{P}(G_{i,I}(X))$, and $h : X \to X$ be a function.*

1. *If $I \in \mathcal{P}_i$, then $K_{hg_{i,I},\mathcal{L}}(X) = K_{hg_{i,I},\widetilde{\mathcal{L}}}(X)$;*
2. *If $I \in \mathcal{P}_i \cap \mathcal{Q}_i$, then $K_{1_X,\mathcal{L}}(X) \subseteq K_{cic,\mathcal{L}}(X) = K_{cic,\widetilde{\mathcal{L}}}(X)$.*

In order to have a simpler notation, the operator $G_{i,I}$ is also denoted by $G(i, I)$. If $I \in \mathcal{P}_i$, then $G(G(i, I), I) = G(i, I)$, and $I \in \mathcal{P}_{G(i,I)}$. Also, if $I \in \mathcal{P}_i \cap \mathcal{Q}_i$, then $I \in \mathcal{P}_{G(i,I)} \cap \mathcal{Q}_{G(i,I)}$.
Let i_1, i_2 be two interior operators on X, and $I_1, I_2 \in \mathcal{I}(X)$. We have the next properties:
If $i_1 \leq i_2$, $I_1 \subseteq I_2$ and $I_1 \in \mathcal{P}_{i_1}$, then $G(i_1, I_1) \leq G(G(i_1, I_1), I_2) \leq G(i_2, I_2)$.
If $I_1 \subseteq I_2$ and $I_1, I_2 \in \mathcal{P}_{i_1}$, then $I_2 \in \mathcal{P}_{G(i_1,I_1)}$.
If $I_1 \subseteq I_2$, $I_1 \in \mathcal{P}_{i_1}$, and $I_2 \in \mathcal{Q}_{i_1}$, then $I_2 \in \mathcal{Q}_{G(i_1,I_1)}$.

Definition 3. *A set $A \subseteq X$ is called i-separated if for all $x \in A$, there exist $u_x \in i(X)$ such that $x \leq u_x$, and $u_x \sqcap u_y = \bot$ for all $x, y \in A$ with $x \neq y$. $\mathcal{S}_i$ is the family of i-separated subsets of X.*

Lemma 1. *Let $f, h : X \to X$ be two functions such that $f(\bot) = \bot$, h is isotone, and $h \leq f(ic)^2$. Let $J \subseteq X$ be a lower set and $I = S_\alpha(h^{-1}(J))$. Then for all $x \in X$ with $x \leq g_{i,I}(x)$, there exist $y, z \in X$ such that $x = y \sqcup z$, $h(y) = \bot$ and $z = \sqcup Z$, where $|Z| \leq \alpha$ and for all $v \in Z$ there is an i-separated set $A_v \subseteq h^{-1}(J)$ such that $v = \sqcup A_v$.*

Theorem 3. *Let $\alpha \geq 2$ be a cardinal number with $\alpha^2 = \alpha$, $J \subseteq X$ a complete ideal, and $f, g, h : X \to X$ three functions such that $f(\bot) = \bot$, $h \leq f(ic)^2$, $1_X \leq g$, $g(J) \subseteq J$ and h is $(g, \mathcal{S}_i)$-Scott-continuous. Then $S_\alpha(h^{-1}(J)) \in \mathcal{I}_\alpha(X) \cap \mathcal{P}_i$.*

We have that i is $(1_X, \mathcal{S}_i)$-Scott-continuous, cic is $(ic, \mathcal{P}(X))$-Scott-continuous, and the functions $(ic)^2$, $(ci)^2$, $i(ci)^2$, $c(ic)^3$, $(ci)^4$ are $(ic, \mathcal{S}_i)$-Scott-continuous. Thus, we get the following results.

Theorem 4. *Let $\alpha \geq 2$ be a cardinal number with $\alpha^2 = \alpha$ and $J \subseteq X$ be a complete ideal.*
Then $S_\alpha(i^{-1}(J)) \in \mathcal{I}_\alpha(X) \cap \mathcal{P}_i$.

Theorem 5. *Let $\alpha \geq 2$ be a cardinal number with $\alpha^2 = \alpha$, and $J \subseteq X$ be a complete ideal such that $cic(J) \subseteq J$. If $h \in \{(ic)^2, (ci)^2, i(ci)^2, c(ic)^3, (ci)^4\}$, then $S_\alpha(h^{-1}(J)) \in \mathcal{I}_\alpha(X) \cap \mathcal{P}_i$.*

Corollary 1. *Let $\alpha \geq 2$ be a cardinal number with $\alpha^2 = \alpha$ and $h \in \{(ic)^2, (ci)^2, i(ci)^2\}$.*
Then $S_\alpha(h^{-1}(\bot)) \in \mathcal{I}_\alpha(X) \cap \mathcal{P}_i$.

Let $\tau \subseteq \mathcal{P}(min)$ be a topology. For $A \subseteq min$, we denote by $int_\tau A$ the interior of A regarding τ.
Let $i : X \to X$ be the function defined by $i(x) = \sqcup int_\tau(min \cap \downarrow x)$; then i is an interior operator.
Let $\alpha \geq 2$ be a cardinal number, $\mathcal{I} \subseteq \mathcal{P}(min)$ an α-ideal with respect to inclusion, and $I = \{\sqcup A \mid A \in \mathcal{I}\}$.
Then $I \in \mathcal{I}_\alpha(X)$, $i(x) \in I \Leftrightarrow int_\tau(min \cap \downarrow x) \in \mathcal{I}$, and $i(X) \cap I = \{\bot\} \Leftrightarrow \tau \cap \mathcal{I} = \{\emptyset\}$.
Therefore, $I \in \mathcal{Q}_i$ if and only if $\tau \cap \mathcal{I} = \{\emptyset\}$.

Theorem 6. *Let d be a metric on min such that (min, d) is a complete metric space, let τ be the metric topology, and $\mathcal{I}$ the σ-ideal of meagre sets. Then $I \in \mathcal{I}_{\aleph_0}(X) \cap \mathcal{Q}_i$.*

Theorem 7. *Let τ be a topology on min such that (min, τ) is a local compact space, and $\mathcal{I}$ the σ-ideal of meagre sets. Then $I \in \mathcal{I}_{\aleph_0}(X) \cap \mathcal{Q}_i$.*

Theorem 8. *Let $\sqsubseteq$ be a partial order on min such that $(min, \sqsubseteq)$ is an algebraic domain, and let τ be the density topology (see [1]). If $\alpha \geq 2$ is a cardinal number and $\mathcal{I}$ the ideal of all α-unions of τ-nowhere dense (or rare) subsets of min, then $I \in \mathcal{I}_\alpha(X) \cap \mathcal{Q}_i$.*

[1] D. Rusu, G. Ciobanu. Essential and density topologies of continuous domains, *Annals of Pure and Applied Logic* 167(9), 726–736, 2016.