On the non-existence of some universal spaces

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Every separable metric space embeds into the Hilbert cube $[0,1]^\omega$ (which is metric), and every countably-based space embeds into the Scott domain $\mathcal{O}(\mathbb{N})$ (which is countably-based). These universal spaces are very convenient to have around, and often used in computable analysis. Schröder has recently shown that there is a universal coPolish space[4], too. Beyond that, we do not see any further universal spaces mentioned in the computable analysis literature.

It would seem particularly desirable to have universal spaces for the countably-based spaces satisfying some separation axioms such as T_1, T_2 or $T_{2.5}$. However, Kihara, Ng and P. showed that there is not even a T_1 countably-based space into which every effective $T_{2.5}$ countably-based space embeds ([2, Theorem 5.17]). As such, hoping for any additional universal spaces for nice classes of countably-based spaces seems pointless.

At first glance, [2, Theorem 5.17] may seem absurd – one might reason that there ought to be countably many **effective** $T_{2.5}$ countably-based spaces, and that a disjoint union should yield a (not necessarily effective) $T_{2.5}$ countably-based space again. However, the result tells us that there are in fact uncountably many $T_{2.5}$ countably-based spaces – and there is no way around this by taking some form of completion[1]. The same holds for effective Hausdorff spaces.

To elucidate this, we consider Plotkin's $\mathbb{T}$, i.e. the space $\{0,1,\bot\}$ (as studied e.g. in [3]). We can show that the effective T_2 countably-based spaces are, up to computable isomorphism, those subspaces of $\mathbb{T}^\omega$ where $x \neq y$ implies $\exists n \ \bot \neq x(n) \neq y(n) \neq \bot$. In particular, there is a single algorithm witnesses the effective Hausdorff condition for all effective T_2 countably-based spaces up to isomorphism. The problem is that correctness of this algorithm relies on a certain sparsity of points.

For the effective $T_{2.5}$ countably-based spaces, we can use the space $\mathbb{W}$ with underlying set $\{\ell, m, r, \bot_\ell, \bot_r\}$ where names for $\bot_\ell$ can always change to names for ℓ or m, and names for $\bot_r$ can always change to names for m or r. We are then looking at subspaces of $\mathbb{W}^\omega$ such that whenever $x \neq y$, then there is some n such that $\{x(n), y(n)\} = \{\ell, r\}$. Up to computable isomorphism, these are exactly the effective $T_{2.5}$ countably-based space. Again, this can be seen as a sparsity notion.

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[1] The completion recently proposed by de Brecht[1] is a completion of subbases, not one of spaces.

References

[1] Matthew de Brecht (2020): *Some Notes on Spaces of Ideals and Computable Topology.* In: Marcella Anselmo, Gianluca Della Vedova, Florin Manea & Arno Pauly, editors: *Beyond the Horizon of Computability – Proceedings of CiE 2020*, Springer, pp. 26–37.

[2] Takayuki Kihara, Keng Meng Ng & Arno Pauly (2019). *Enumeration degrees and non-metrizable topology.* arXiv:1904.04107.

[3] Arno Pauly & Hideki Tsuiki (2016). *T^ω-representations of compact sets.* arXiv:1604.00258.

[4] Matthias Schröder (2019). *On maximal Co-Polish Spaces.* CCA 2019.