Computable profinite groups and randomness

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This is joint work on an ongoing project with Andre Nies.

1 A theorem

We shall discuss the following

Theorem 1 *(Fouché, Nies).* `Based on research as can be found in the book ''Field arithmetic'' by Fried and Jarden.`

- *Let G be the absolute Galois group of the field $\mathbb{Q}$ of rational numbers, topologised by the standard Krull topology.*
- *Write μ for the Haar probability measure on G.*

  Given a computable representation of G, μ and an element σ of G which is Kurtz-random relative to μ, write $[\sigma]_n$ for the topological normal closure of σ which is the smallest topological and normal subgroup of G that has σ as an element.

  Then $[\sigma]_n$ is isomorphic to $\hat{\mathbb{F}}_\omega$, the free profinite group on countably many generators.

  Moreover, the fixed field L_σ corresponding via Galois duality to the normal closed group $[\sigma]_n$, is pseudo-algebraically closed.

2 Explication of terminology

1. We fix an algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$. Then algebraically, the group G is given by all the automorphisms of the field $\overline{\mathbb{Q}}$ that keeps $\mathbb{Q}$ fixed. It can be shown that

$$G \simeq \varprojlim_{N} G/N,$$

   where N ranges over all the normal subgroups of finite index of G. The topological structure of G is inherited by embedding $\varprojlim_{N} G/N$ in the product of all the G/N

with N normal and of finite index in G. It follows that G is a compact Hausdorff group. Being a projective limit of finitely many finite groups, one can embed G as a closed subgroup of the Baire space ω^ω. Using ideas going back to Kronecker on effective Galois theory over the rationals we can embed G as a Π^0_1 subspace of ω^ω. The Haar measure on G is computable relative to this representation of G in Baire space.

2. Recall that a field L is *pseudo-algebraically closed* if every absolutely irreducible algebraic variety over L will have a point all coordinates of which belong to L.

3. An element σ of G is *Kurtz-random* if it belongs to every Σ^0_1 (note lightface!) set of μ-measure 1.

4. Finally,
$$\hat{\mathbb{F}}_\omega \simeq \varprojlim_N \mathbb{F}_\omega / N,$$
where N ranges over the normal subgroups of finite index in the free group $\mathbb{F}_\omega$ on countably many generators.

3 Context of research

- A case can be made that profinite groups are absolute Galois groups of fields (Waterhouse). Such groups have a natural (Krull) topology which can be expressed in the language of profinite topologies.
- This renders the groups topologically as compact Hausdorff spaces which are totally disconnected. These groups G have unique Haar measures μ with $\mu(G) = 1$.
- In this project, we consider μ-almost sure properties of elements of G and explore the algorithmically random complexity of almost sure properties relative to μ where the group G and μ have computable representations.
- In this talk, the cases when G is the absolute Galois group of a computable Hilbertian field with splitting algorithm, a finite field, or an effective pseudo-algebraically closed field will be discussed. (The notions will be explained during the talk). If time permits we will also explore phenomena which are essentially Schnorr random.